

Information Divergences

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Let $f \in L^2(\Omega)$ be an image possibly corrupted by noise, defined on a bounded open set $\Omega \subset \mathbb{R}^2$. We may assume $f(x) \in [0, 1]$. In image processing via variational methods, one is interested in decomposing f into u , and v , by an energy minimization. Here u represents the “true features” in f , and v represents the “noisy components” in f .

In 1989, D. Mumford and J. Shah introduced a model which decomposes f corrupted by additive gaussian noise into $u + v$, via

$$\inf_{u, \Gamma} \{J^{MS}(u, \Gamma) = \int_{\Omega \setminus \Gamma} |\nabla u|^2 + \mu \int_{\Omega} (f - u)^2 + \lambda |\Gamma|\},$$

where μ , and λ are weighting parameters which need to be chosen effectively, Γ is a piecewise smooth curve in \mathbb{R}^2 , and $|\Gamma|$ denotes the one dimensional Hausdorff measure of Γ .

In 1992, L. Rudin, S. Osher, and E. Fatemi proposed a simpler model, where they impose $u \in BV(\Omega)$, and $v \in L^2(\Omega)$. Here $BV(\Omega)$ is a space of functions of bounded variations. The image f in this model is decomposed into $u + v$ by an energy minimization,

$$\inf_u \{J^{ROF}(u) = \int_{\Omega} |\nabla u| + \lambda \int_{\Omega} (f - u)^2\},$$

referred to as the ROF model.

However, as Y. Meyer points out in [?], if one decomposes f into $u + v$ using the ROF model, the v component is not purely noise, it contains some image features that are supposed to be in u . For example, if f is a characteristic function of a disk, with no noise, then for any $\lambda < \infty$, the ROF model will not give $v = 0$.

From this motivation, other models have been introduced. For example, in 2002, L. Vese and S. Osher proposed to decomposed f into $u + v$ by an

energy minimization,

$$\inf_{u \in BV(\Omega), \vec{g} \in L^p(\Omega)} \{J^{VO}(u, \vec{g}) = \int_{\Omega} |\nabla u| + \mu \|f - u - \text{div} \vec{g}\|_{L^2(\Omega)}^2 + \lambda \|\vec{g}\|_{L^p(\Omega)}\},$$

where $v = \text{div} \vec{g}$, and $|\vec{g}| = \sqrt{g_1^2 + g_2^2}$.

Other models include:

1. S. Osher, A. Sole, L. Vese (2002):

$$\inf_u \{J^{OSV}(u) = \int_{\Omega} |\nabla u| + \lambda \int_{\Omega} |\nabla(\Delta^{-1}(f - u))|^2\}.$$

Here $u \in BV(\Omega)$, and $v \in H^{-1}(\Omega)$.

2. T. Chan, and S. Esedoglu (2004):

$$\inf_u \{J^{CS}(u) = \int_{\Omega} |\nabla u| + \lambda \int_{\Omega} |f - u|\}.$$

In this model, $u \in BV(\Omega)$, and $v \in L^1(\Omega)$.

The models described above are used to denoise additive gaussian noise of mean zero. Images with other types of noise, say multiplicative noise of mean one, where $f = uv$ and $\int_{\Omega} v = 1$, are denoised using algorithms such as the ROF model with a different fidelity term:

$$\inf_u \{J^{ROF}(u) = \int_{\Omega} |\nabla u| + \lambda \int_{\Omega} (\frac{f}{u} - 1)^2\}.$$

We also consider an energy minimization inspired by [?] to model poisson noise,

$$\inf_{u \in BV(\Omega)} \{J(u) = \int_{\Omega} |\nabla u| + \lambda \int_{\Omega} (u - f \log u)\}.$$

In this paper, we introduce a statistical model to capture the similarities in images with various types of noise, and to compare the recovered image of different reconstruction models in the literature. The types of noise considered are additive gaussian, multiplicative, salt and pepper, poisson, and cracks. Currently, our focus is on additive gaussian noise.

Description of our model:

Suppose f is a discrete image of $N \times M$ pixels. We can view f as being an element of $\mathbb{R}^{N \times M}$. However, for natural images, the dimension is far smaller than NM . Therefore, we define a filter, F , which decreases the dimension of the images,

$$F : \mathbb{R}^{N \times M} \rightarrow \mathbb{R}^{n \times m},$$

where $n \ll N$, and $m \ll M$.

Given an image $f(x) \in [0, 1]$, define its histogram h_f by

$$h_f(i) = |\{x : f(x) \in [i\Delta h, (i+1)\Delta h]\}|, \quad i = 0, 1, \dots,$$

where $\Delta h = \frac{1}{255}$. The cumulative distribution function is then defined by

$$H_f(k) = |\{i : h_f(i) \leq k\}|, \quad k = 0, 1, \dots$$

Now, let T be the true image, N be the true image T corrupted by additive noise, and R be the recovered image (from applying one of the above denoised algorithms). We now will quantify how good the recovered image R is, according to three different metrics:

1. $\|H_{F(T)} - H_{F(R)}\|_{L^\infty} + \|H_{F(N-R)} - H_{F(N-T)}\|_{L^1}$.
2. $\inf_{\sigma} \{ \|H_{F(H-R)} - H_{F(\eta(\sigma))}\|_{L^1} \} + \|H_{F(T)} - H_{F(R)}\|_{L^\infty}$, where $\eta(\sigma)$ is a purely gaussian random noise image of standard deviation σ .
3. $\sigma^* + \|H_{F(R)} - H_{F(T(\sigma^*))}\|_{L^2}$, where σ^* minimizes $\inf_{\sigma} \{ \|H_{F(H-R)} - H_{F(\eta(\sigma))}\|_{L^1} \}$, and $T(\sigma^*)$ is the true image corrupted by additive gaussian noise of standard deviation σ^* .

Different filters, F , will be used to greatly distinguish between clean and noisy images. Also, other forms of metrics will be considered for other types of noise.

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References

- [1] T. Chan, and S. Esedoglu, "Aspect of Total Variation Regularized L^1 Function Approximation", UCLA CAM Report 04-07, Feb. 2004.
- [2] M. Green, "Statistics of Images, the TV Algorithm of Rudin-Osher-Fatemi for Image Denoising and an Improved Denoising Algorithm", UCLA CAM Report 02-55, Oct. 2002.
- [3] Y. Meyer, "Oscillating Patterns in Image Processing and NonLinear Evolution Eautions", University Lecture Series, Vol. 22, AMS 2002.
- [4] D. Mumford, and J. Shah, "Optimal Approximation by Piecewise Smooth Functions and Associated Variational Problems", Comm. Pure Appl. Math., Vol. 42, pp. 577-685, 1989.
- [5] L. Rudin, S. Osher, and F. Fatemi, "Nonlinear Total Variation Based Noise Removal Algorithm," Physica D, Vol 60, pp. 259-268, 1992.
- [6] S. Osher, A. Sole, and L. Vese, "Image Decomposition and Restoration Using Total Variation Minimization and the H^{-1} Norm", UCLA CAM Report 02-57, Oct. 2002.
- [7] L. Vese, and S. Osher, "Image Denoising and Decomposition with Total variation Minimization and Oscillatory Functions", J. Math. Imag. Vision, Vol. 20, pp. 7-18, 2004.